

# The Pauli Exclusion Principle and $SU(2)$ Versus $SO(3)$ in Loop Quantum Gravity

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## ABSTRACT

Recent attempts to resolve the ambiguity in the loop quantum gravity description of the quantization of area has led to the idea that  $j = 1$  edges of spin-networks dominate in their contribution to black hole areas as opposed to  $j = 1/2$  which would naively be expected. This suggests that the true gauge group involved might be  $SO(3)$  rather than  $SU(2)$  with attendant difficulties. We argue that the assumption that a version of the Pauli principle is present in loop quantum gravity allows one to maintain  $SU(2)$  as the gauge group while still naturally achieving the desired suppression of spin-1/2 punctures. Areas come from  $j = 1$  punctures rather than  $j = 1/2$  punctures for much the same reason that photons lead to macroscopic classically observable fields while electrons do not.

## I. INTRODUCTION

The recent successes of the approach to canonical quantum gravity using the Ashtekar variables have been numerous and significant. Among them are the proofs that area and volume operators have discrete spectra, and a derivation of black hole entropy up to an overall undetermined constant [1]. An excellent recent review leading directly to this paper is by Baez [2], and its influence on this introduction will be clear.

The basic idea is that a basis for the solution of the quantum constraint equations is given by *spin-network* states, which are graphs whose edges carry representations  $j$  of  $SU(2)$ . To a good approximation, the area  $A$  of a surface which intersects a spin network at  $i$  edges, each carrying an  $SU(2)$  label  $j$  is given in geometrized units (Planck length equal to unity) by

$$A \approx \sum_i 8\pi\gamma\sqrt{j_i(j_i+1)} \quad (1)$$

where  $\gamma$  is the Immirzi-Barbero parameter [3]. The most important microstates consistent with a given area are those for which  $j$  is as small as possible, which one would expect to be  $j_{min} = 1/2$ . In this case, each contribution to the area corresponds to a spin  $j = 1/2$  which can come in two possible  $m$  values of  $\pm 1/2$ . For  $n$  punctures, we have  $A \approx 4\pi\sqrt{3}\gamma n$  and entropy  $S \approx \ln(2^n) \approx \frac{\ln(2)}{4\pi\sqrt{3}\gamma} A$ .

Now looking outside loop quantum gravity for help, we can use Hawking's formula [4] for black hole entropy  $S = A/4$  to get  $\gamma = \frac{\ln(2)}{\pi\sqrt{3}}$  and the smallest quantum of area is then  $8\pi\gamma\sqrt{\frac{1}{2}(\frac{1}{2}+1)} = 4\ln(2)$ . Physically this is very nice as it says that a black hole's horizon acquires area, to a good approximation, from the punctures of many spin network edges, each carrying a quantum of area  $4\ln(2)$  and one "bit" of information – a vindication of

Wheeler’s “it from bit” philosophy [5].

Bekenstein’s early intuition [6] that the area operator for black holes should have a discrete spectrum made of equal area steps (something not really quite true in loop quantum gravity in full generality) was followed by Mukhanov’s observation [7] that the  $n^{th}$  area state should have degeneracy  $k^n$  with steps between areas of  $4\ln(k)$  for  $k$  some integer  $\geq 2$  in order to reproduce the Hawking expression  $S = A/4$ . For  $k = 2$  one would have the  $n^{th}$  area state described by  $n$  binary *bits*.

On the other hand, Hod [8] has argued that by looking at the quasinormal damped modes of a classical black hole one should be able to derive the quanta of area in a rather different way. The basic idea is to use the formula  $A = 16\pi M^2$  relating area and mass of a black hole to get  $\Delta A = 32\pi M \Delta M$  for the change in area accompanying an emission of energy  $\Delta M$ . Nollert’s computer calculations [9] of the asymptotic frequency  $\omega$  of the damped normal modes gave  $\omega \approx 0.4371235/M$ , so setting  $\omega = \Delta M$  one finds  $\Delta A \approx 4.39444$ . It is tempting then to conclude that perhaps  $\Delta A = 4\ln(3)$ . Motl [10] later showed that this is indeed correct, and not just a fortuitous numerical coincidence.

Since then, Dreyer [11] has pointed out that one might well expect  $\Delta A \approx 4\ln(3)$  instead of  $\Delta A \approx 4\ln(2)$  if the spin network edges contributing to the area of a black hole didn’t carry  $j = 1/2$ , but rather  $j = 1$ . In this case  $j_{min}$  would be 1 rather than  $1/2$ , there would be three possible  $m$  values, and area elements would be described not by binary “bits”, but by trinary “trits”. (See also [12]). This also suggests that perhaps the correct gauge group is not  $SU(2)$  but  $SO(3)$ , although this could complicate the inclusion of fermions in the theory.

Corichi has recently argued [13] that one might arrive at the conclusion that  $j_{min} = 1$  by

suggesting that one should think of a conserved fermion number being assigned to each spin-1/2 edge. Adding or losing an edge's worth of area would have to mean that at some point a spin-1/2 edge would be essentially dangling in the bulk (*i.e.* not imbuing the horizon surface with area) and this should not be allowed. If edges carried  $j = 1$  one could imagine coupling the edge to a fermion-antifermion pair and this would locally solve the fermion number problem. This is quite appealing as one might then think of the loss of an element of area with accompanying fermion-antifermion production in Hawking radiation as the detachment of a spin-1 edge from the horizon which then couples to an  $f\bar{f}$  pair. As Corichi [13] points out:

*“the existence of  $j = 1/2$  edges puncturing the horizon is not forbidden ..., but they must be suppressed. Thus, one needs a dynamical explanation of how exactly the entropy contribution is dominated by the edges with the dynamical allowed value, namely  $j = 1$ .”*

## II. THE EXCLUSION PRINCIPLE

The point of this essay is to suggest that one might want to assume that a version of the spin-statistics theorem (or, equivalently, the Pauli exclusion principle) applies to loop quantum gravity. More precisely, it could be the case that no more than two punctures of  $j = 1/2$ , each with differing  $m$  values, may puncture a given surface. In this case, the dominance of  $j = 1$  punctures (even though  $j = 1/2$  is allowed) is very natural: if only a maximum of two spin-1/2 edges can puncture any surface then for large numbers of punctures one would have an effective  $j_{min} = 1$  despite the gauge group being  $SU(2)$ .

It is not immediately obvious what (if any) sort of extension of the exclusion principle

for particles in spacetime carrying  $SU(2)$  representation labels should apply to spin network edges carrying  $SU(2)$  representation labels. In quantum mechanics, the spin-statistics theorem is simply a postulate, since, as Dirac [14] puts it “to get agreement with experiment one must *assume* that two electrons are never in the same state” (my italics). The same sort of reasoning could be applied here, but it is also possible to make a stronger case for the idea.

The spin-statistics theorem as usually formulated and proven (to the extent that one rigorously proves anything in quantum field theory!), is, of course, for matter fields in a background spacetime usually assumed to be flat. For bosonic fields one makes the usual expansion of plane waves in terms of creation and annihilation operators and then demands that at spacelike separations the field operators should commute. This is certainly physically reasonable and is meant to capture the appropriate notion of causality, for one would not expect physics at some point in spacetime to be affected by what goes on outside its light cone. For fermion fields one is led to look at anticommutators instead in order to reproduce the experimentally observed fact you don’t find two identical fermions in the same state. Anticommutators were basically pulled out of thin air by Jordan [15] and any relation now to causality is obscured unless one assumes that the fermion fields are Grassman-valued. Most modern quantum field theoretic proofs of the spin-statistics theorem are essentially carried out by seeing what happens if one assumes the wrong choices of commutation or anticommutation relations and finding that things don’t work out. For example, quantizing the Dirac field with commutators gives an unstable vacuum. A rather comprehensive review of the history and literature is in [16].

Here we don’t have particles, but rather edges of spin networks, and we don’t really

have spacetime either except in some approximation, so we have to try to think about the spin-statistics theorem in a slightly less spacetime-bound way. That being said, the fact that the edges are meant to puncture a spacelike surface like the horizon of a black hole is encouraging as it might make some sense to think of the punctures are being spacelike separated and possibly subject to commutation or anticommutation relations. In addition, the spin-statistics association is strongly combinatorial in flavour and seems natural in a spin-network context.

For a surface punctured by spin network edges I want to argue that one should consider an amplitude which returns to its original value, up to a phase, upon the exchange of two spin-1/2 (and thus identical, indistinguishable) punctures. If making the exchange twice leads to the identity<sup>1</sup>, one then needs merely to choose a sign, and -1 seems at least as natural as +1. This argument can be sharpened in the following way:

Let us consider the configuration space of  $n$  spin- $j$  non-coincident *identical* punctures and see what we can expect on fairly general grounds. First of all, as shown long ago by Laidlaw and DeWitt [18], that phases for propagators in a multiply connected configuration space must form a scalar unitary representation of the fundamental group. For us, the configuration space is the set of  $n$  non-coincident points on a sphere, and this means that the phases must form a scalar unitary representation of the permutation group. That then limits the possible choice of statistics to Bose statistics (no phase change under permutations) or Fermi statistics (change of sign for any odd permutation). As it stands, this is just a

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<sup>1</sup>It is interesting to consider the possibility of more exotic braid or anyon-like statistics if one would have to keep track of how one edge moved around another, but this is beyond the scope of this essay, but see [17].

statement about possible statistics and has nothing to do with rotations,  $SU(2)$ ,  $SO(3)$ , or even physical space, but it does fix the possible choices we can make.

Now to argue for Fermi statistics for odd half-integer  $j$  punctures and Bose statistics for integer  $j$  punctures we cannot use the usual QFT arguments. We have no suitable creation and annihilation operators, no background spacetime, no plane wave expansion, *etc.* In particular, simple arguments based on identifying an exchange as a composition of physical rotations (*i.e.* [19]) seem inapplicable as we don't have a background space in which to rotate, as do arguments based on extended kink-like objects (*i.e.* [20]) since the punctures are meant to be points. Related approaches such as those of Balachandran *et al.* [22], Tseuschner [21], Berry and Robbins [23], and many others in [16] all seem difficult to apply here.

What we need to do is look for a proof (or at least an argument) for the spin-statistics relation that is not rooted in a prior concept of physical space. One possibly applicable route is to just go directly to the configuration space and use ideas from geometric quantization [24], which I now do, following rather beautiful arguments of Anastopoulos [25].

In geometric quantization one starts with a classical phase space  $\Gamma$  with its associated symplectic form  $\Omega$ . Quantum mechanics and complex numbers enter via the *prequantization* of  $(\Gamma, \Omega)$  given by a  $U(1)$  fiber bundle  $(Y, \Gamma, \pi)$  with total space  $Y$ , base space  $\Gamma$  and projection map  $\pi : Y \rightarrow \Gamma$ . A  $U(1)$  connection  $\omega$  in  $Y$  is required to satisfy  $d\omega = \pi_*\Omega$ . This then requires the integral of  $\Omega$  over any surface  $\int \Omega$  to be a multiple of  $2\pi$ . (The line integral of  $d\omega$  is clearly zero around any closed path  $\gamma$ . This corresponds to trivial holonomy  $\exp(i \int_\gamma \omega)$ , which can also be calculated by Stokes theorem from the integral of  $\Omega$  on any surface  $\sigma$  bounded by such a closed path and the result follows.) Sections of the  $U(1)$  bundle

(suitably completed) then form the Hilbert space for the quantum system corresponding to the classical  $(\Gamma, \Omega)$ .

Now consider the quantization of the sphere  $S^2$ , coordinatized  $S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$  with the symplectic form

$$\Omega = \frac{1}{2} s \epsilon_{ijk} x^i dx^j \wedge dx^k \quad (2)$$

and a symplectic action of  $SO(3)$  on  $S^2$  where  $SO(3)$  acts on the  $x^i$  in the usual way by its defining representation and obviously leaves  $\Omega$  invariant. Each choice of  $s$  gives a different symplectic manifold, and the requirement that  $\Omega$  be integrable requires that  $s = n/2$  with  $n$  an integer. In this way  $s$  corresponds to the usual notion of spin in quantum mechanics. Note that so far there is no explicit identification of the  $x^i$  with spacetime directions – they just happen to define the coordinates on an abstract  $S^2$ .

An explicit realization of the  $U(1)$  bundle is provided by the Hopf map  $\pi(\xi)^i = \bar{\xi} \sigma^i \xi$  which is defined in terms of 2-component spinors  $\xi$  normalized to length 1 by  $\bar{\xi} \xi = 1$  with  $\sigma^i$  the usual Pauli matrices. Note that the  $\xi$  with this normalization lie on  $S^3$ . There is also the natural connection  $\omega = -i \bar{\xi} d\xi$  and a  $U(1)$  action on the fibers which is just multiplication by a phase. This  $U(1)$  action will be very important in what follows.

The Hopf map clearly takes one from the  $U(1)$  bundle over  $S^2$  down to  $S^2$  since  $\pi(\xi)^i$  is obviously real and invariant under  $\xi \rightarrow u\xi$  for any  $u = \exp(i\theta)$  in  $U(1)$ . While a full treatment for arbitrary  $s$  can be found in [25], let us just consider the case of  $s = 1/2$ . In this case we have  $U(1)$  operations corresponding to the two square roots of unity  $\pm 1$ , both of which correspond to the identity element of  $SO(3)$ .

We can now consider the bundle whose total space is the set of orbits of  $S^3$  under the



two  $U(1)$  actions of multiplication by  $\pm 1$  and the same projection map  $\pi$ . This is our prequantization.  $SU(2)$  actions on  $S^3$  correspond to  $SO(3)$  actions on  $S^2$ , and we pick up a factor of -1 for a  $2\pi$  rotation. We can now think of an  $s = 1/2$  state as a point on the sphere  $S^2$ , accompanied by this sign change for  $2\pi$  rotations. (Much the same argument goes through in general for higher spins in a similar fashion: for spin  $n/2$  we have  $n$  roots of unity  $\exp(2\pi ir/n)$  with  $r$  ranging from 0 to  $n - 1$  and the orbits of  $S^3$  under these  $U(1)$  actions is the prequantization. As one might expect, there is a sign change under  $2\pi$  rotations for half-integer spins and none for integer spins.)

Now consider two classical phase spaces  $\Gamma_i$  ( $i = 1, 2$ ) with symplectic forms  $\Omega_i$ . The combined system then has classical phase space  $\Gamma_1 \times \Gamma_2$  and symplectic form  $\Omega_1 \oplus \Omega_2$ . Repeating the same construction, and assuming the two  $\Gamma_i$  are the same and both  $S^2$ , we can ask what it would take to effect an *exchange* carrying  $(\xi_1, \xi_2)$  to  $(\xi_2, \xi_1)$ . This can be accomplished by two  $SO(3)$  rotations, one of the first  $\Gamma_1 (= S^2)$  on which  $\xi_1$  lies, and one on the second  $\Gamma_2 (= S^2)$  such that the  $\xi_1 \rightarrow \xi_2$  and  $\xi_2 \rightarrow \xi_1$ . If we believe that  $\xi_1$  and  $\xi_2$  are *indistinguishable*, however, we should really think of them as two points on the *same*  $S^2$ . That means the rotation that exchanges them should act along the same orbit of the  $SO(3)$  action. The net result then is a rotation of  $2\pi$  and one has a sign of  $\pm 1$  depending on the parity of  $n$  just as one would want for a spin-statistics theorem. In other words, the intuitive arguments referred to earlier which are used to argue for the spin-statistics theorem following from the equivalence of an exchange with a rotation of  $2\pi$  in physical space can also be used here even though the  $S^2$  was just introduced as a way of dealing with  $SO(3)$  and, through geometric quantization,  $SU(2)$ .

In other words, I would argue that even though one might colloquially speak of the

edges as carrying “spins”, knowing full well that this is really a way of saying “ $SU(2)$  representation labels with no obviously necessary connection to spin of elementary particles or irreducible representations of the rotation group in physical space”, in fact it *does* make sense to think of them as physical spins and argue for a spin-statistics theorem. In this sense the spin-statistics theorem might better be thought of as a sort of “( $SU(2)$  representation label)-(sign change or not on exchange)” theorem.

In loop quantum gravity this leads then to a picture in which a reasonably large black hole *can* get area contributions from spin-1/2 (and spin-3/2, spin-5/2, *etc.*) punctures, but these are always very small compared to the enormous number of  $j = 1$  edges. The value  $j = 1$  is the lowest value of  $j$  contributing nonzero area not being severely limited by Fermi-Dirac statistics, and able to appear arbitrarily often. This can then make it look like we’re dealing with  $SO(3)$  rather than  $SU(2)$ .

### III. CONCLUSIONS

In a sense, the question of  $SU(2)$  vs.  $SO(3)$  in loop quantum gravity could be very much like one that we face in everyday physics. Integer spin particles, which fall into  $SO(3)$  representations, obey Bose-Einstein statistics and gregariously bunch together to give large macroscopically observable fields such as electromagnetic fields. Half-integer spin particles do not. We could well be excused for thinking that the symmetry group of our world under rotations was  $SO(3)$  rather than  $SU(2)$ . Indeed, until the discovery of spin, it did appear that physical rotations were always elements of  $SO(3)$ . The need for  $SU(2)$  was, in many ways, a surprise!

It may be hard to find direct experimental evidence of these ideas, but it is at least possible to make some predictions. For example, the  $SU(2)$  theory with the exclusion principle proposed here will give both:

- a) what seems to be the correct result for large black holes, with areas well-described by values which go up *in steps of*  $4 \ln(3)$ ; and
- b) the possibility of simultaneously admitting areas *as small as*  $4 \ln(2)$ .

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